Apparent Tensorial Conductivity of Layered Composites

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A lamination theory is proposed to treat a multilayered composite material as an anisotropic but homogeneous equivalent material. The conductivity tensor and the governing differential equations of the equivalent material are derived. A sample heat conduction problem for a laminated cylindrical body is solved by the present approach and results compared favorably with the exact solution.

Nomenclature

K = conductivity
K_{ij} = conductivity tensor
q_i = heat flux in the direction of x_i
R_{ij} = resistivity tensor
T = temperature
V = volume fraction
ϕ = dimensionless temperature defined in Eq. (21)
η = dimensionless temperature defined in Eq. (27)
δ_{ij} = Kronecker delta

Superscripts
"p" = pth layer or component
$(\bar{\ })$ = dimensionless quantities defined in Eqs. (21) and (27).

Introduction

In recent years, composite materials have been developed for structural application to aerospace vehicles under extreme environmental and loading conditions. The most commonly used systems, such as boron-epoxy, boron-aluminum, and graphite-epoxy are constructed on a production basis.[1] With the rapid growth of the composite technology, the mechanical behavior of the composite has been studied extensively. The study of heat conduction in a composite, however, is quite limited in the literature.

One of the most important problems of heat conduction in a composite is to determine the thermal conductivities of the composite from given thermal conductivities of its components. The problem often becomes complicated because of the anisotropic behavior of the composite. In some cases, the effect of this material anisotropy is very strong. For example, the conductivity of a layered graphite in the direction parallel to the layer may be two orders of magnitude higher than that in the directions normal to it.[2] Most of the past work, however, considers the cases where the components of the composite are isotropic. An extensive review of analytical methods in treating these cases is available in Russian literature.[3] More recently, Boley[4] has made a survey of heat conduction in solids in which the subject of the composite material is included. There are 16 papers under this subject. Among them, Springer and Tsai[5] proposed an anology between the elastic shear loading and heat transfer for determination of the conductivity of a unidirectional composite. Applying the method of long waves, Behrens[6] calculated the conductivity of a composite with orthorhombic symmetry. Experimental determination of conductivity of the composites made of metal fibers randomly distributed in an epoxy matrix was performed by D'Andrea and Ling.[7] It was found in this case that the conductivity is controlled by probabilistic, not deterministic, laws.

For a composite of alternating layers of two isotropic, homogeneous materials, Weng and Koh[8] derived a set of heat conduction equations and boundary conditions based on variational principles. No solutions, however, have been presented. The set of equations include two equations with two coupled variables, namely temperature and a new variable that accounts for local variation of temperature. Both equations have higher order terms as compared to the classical conduction equation for anisotropic media. This situation prevents us from comparing the formulation of our present work.

A typical example of the layered anisotropic composite is the filamentary composite that is made up of many thin layers or laminae. One lamina of a filamentary composite consists of one row of parallel filaments surrounded by the matrix. Two such laminae are shown in Fig. 1. The laminae are stacked with various orientations of the filament direction to obtain a layered composite which has the desired stiffness or strength properties.

This paper introduces a lamination theory that treats a multilayered composite as an anisotropic but homogeneous equivalent material, and derives an apparent conductive tensor for the equivalent material in terms of properties of the constituent layers. The governing differential equation of heat conduction can then be written in terms of this conductivity tensor, and boundary value or initial value problems can be solved. These equations for the homogeneous equivalent material are usually easier to solve as compared to problems of the original nonhomogeneous multilayered material.

In what follows, we shall describe in detail the assumptions and techniques involving certain averaging processes used in the present theory. A comparison of the results with exact solutions for the case of a cylindrical sector is finally presented.

Analysis

In an isotropic homogeneous medium, heat conduction is described by Fourier's law in which the heat flux is proportional to the temperature gradient, the constant of proportionality being defined as the thermal conductivity. Mathematically, it is written as

$$-q_i = K(\partial T/\partial x_i) \quad (i = 1, 2, 3) \tag{1}$$

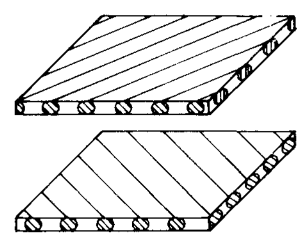

Fig. 1 Layers of a filamentary composite.

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where the conductivity K in this expression is a scalar quantity, which depends only on the thermodynamic coordinates. Because of the gradient terms appearing in the right-hand side of the equation, one observes that the direction of heat flux is normal to the isothermal surfaces. In the case of an anisotropic medium, the conductivity depends on a preferred direction, one assumes,[9]

$$-q_i = K_{ij}\frac{\partial T}{\partial x_j} \quad (i,j=1,2,3) \tag{2a}$$

The conductivity K_{ij} now becomes a second rank tensor. The heat flux vector as seen from Eq. (2a) will no longer be perpendicular to the isothermal surfaces of the medium. Equation (2a) is referred as general Fourier's law from which the conduction equation for a homogeneous, anisotropic medium can be written as

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_i}\left[K_{ij}\frac{\partial T}{\partial x_j}\right] + s \quad (i,j=1,2,3) \tag{2b}$$

where s represents internal heat generation per unit volume.

In treating the problems involving the anisotropic medium, it may be sometimes more convenient to write the temperature gradient in terms of the components of the heat flux

$$-\partial T/\partial x_i = R_{ij}q_j \quad (i,j=1,2,3) \tag{3}$$

where the quantity R_{ij} is called resistivity tensor. To relate the conductivity tensor with the resistivity tensor, one uses the following relation in matrix form

$$[K][R] = [1] \tag{4}$$

Equation (4) states that the multiplication of conductivity matrix to the resistivity matrix is equal to the identity matrix. Using this relation, the elements of one matrix can be computed in terms of the elements of the other.

The conductivity tensor appearing in Eq. (2a) can be applied to a layered composite material if it is smoothed into a homogeneous but anisotropic medium. We shall now outline a lamination theory that is used to obtain the equivalent conductivity tensor of the smoothed composite in terms of the properties of its components. For n-repeated layer composite whose components are allowed to be triclinic, i.e., general anisotropic without any symmetry,[9] we start from the following basic assumptions:

1) The basic element in obtaining Fourier's law of conduction is generalized to include more than one layer. The temperature gradients and heat flow vectors are constant in each layer within the element. This assumption is good if the thickness of each layer is small compared with the characteristic dimension of the composite.

2) Those quantities in different layers within an element that are not equal, average values are taken as the equivalent ones for the smoothed composite.

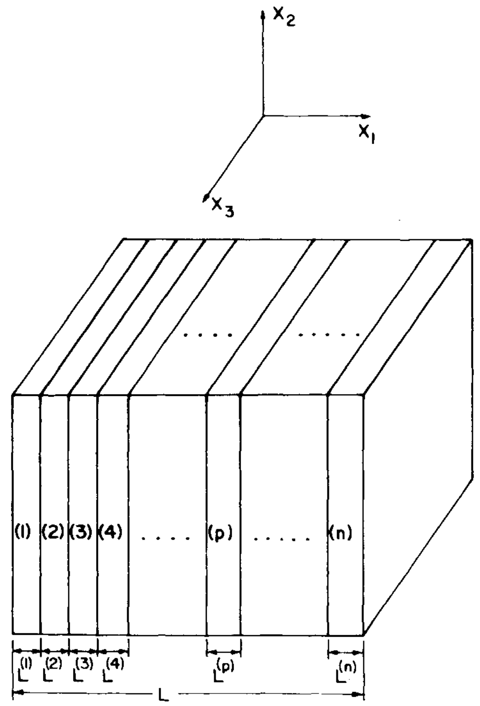

Fig. 2 Volume element of a layered composite material.

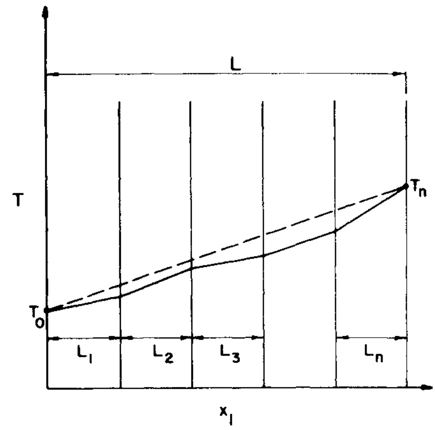

Fig. 3 Average temperature gradient of a smoothed composite.

With the conductivity tensor thus obtained, one can solve field conduction, either boundary value or initial value problems, to give the temperature and heat-flux distribution for the smoothed composite. The proposed theory is then applied to calculate the temperature and heat-flux distribution for each layer of the originally unsmoothed composite since it can be seen later that both temperature gradient and heat flux in each layer can be expressed as functions of the temperature gradient and the heat flux of the smoothed composite.

To develop expressions for the equivalent conductivity tensor of the smoothed composite in terms of the properties of its components, we consider a three-dimensional composite element with n-layered components shown in Fig. 2, each layer being anisotropic; i.e., all $K_{ij}^{(p)}$ are in general different. The superscript "p" denotes the pth layer component with $p = 1,2,3,\ldots,n$. The plane of the layers is shown in x_2x_3 plane. Figure 2 shows only one section of the composite. The composite under consideration is actually made of repeated sections.

Based on our first assumption, we consider the thickness of the layers, L_1, L_2, $L_3 - L_n$ to be so thin that within each layer of the small element, the heat flux, $q_i^{(p)}$ and the temperature gradient, $\partial T^{(p)}/\partial x_i$ are uniform. As a consequence of the uniform heat flux, it is implied that the heat flux of one layer in the x_1 direction is equal to the heat flux of the next layer in that direction, or $q_1^{(1)} = q_1^{(2)} = q_1^{(3)} = \cdots = q_1^{(n)}$ which is in turn equal to the heat flux of the composite q_1 (quantities without superscript denote those of the composite). The relations obtained in this way will satisfy the condition of continuous heat fluxes across each interface. Likewise, the temperature gradients in the x_2 and x_3 direction are considered to be the same, or

$$\partial T^{(1)}/\partial x_i = \partial T^{(2)}/\partial x_i = \partial T^{(3)}/\partial x_i = \cdots = \partial T^{(n)}/\partial x_i, \quad i = 2,3$$

which are in turn equal to the temperature gradients of the composite, $\partial T/\partial x_i$, $i = 2,3$.

Next, the heat flux in the x_2 and x_3 direction and the temperature gradient in the x_1 direction of each layer are in general different. To obtain the heat flux and temperature gradient for the composite, we propose an averaging method expressed in the form

$$\psi = \sum_{p=1}^{n} V^{(p)}\psi^{(p)} \tag{5}$$

where ψ may refer to either q_i $(i = 2,3)$ or $\partial T/\partial x_1$ and the volume fraction $V^{(p)}$ is defined as

$$V^{(p)} = L^{(p)}/L \quad (p = 1,2,\ldots,n) \tag{6}$$

$L^{(p)}$ and L in this expression denotes the thickness of the pth layer and the composite, respectively (see Fig. 2). Now, if Eq. (5) is used to calculate heat flux of the composite q_i $(i = 2, 3)$, the equation will then imply that the total heat flow of the composite in the i direction $(i = 2, 3)$ is equal to the sum of the total heat flow of all components in that direction. Likewise, if Eq. (5) is used to obtain $\partial T/\partial x_1$ of the composite, $\partial T/\partial x_1$ then represents the average value of all $\partial T^{(p)}/\partial x_1$ with the condition of matching temperature at each interface. To explain this situation, let the solid lines shown in Fig. 3 be the temperature profile along x_1 coordinate. The temperature profile of the smoothed composite is approximated by the dotted line T_0T_n. The apparent temperature gradient of the composite element along x_1 coordinate is then given by $(T_n - T_0)/L$ which is equal to the weighted sum of the temperature gradient of each layer along the same coordinate, i.e.,

$$\sum_{p=1}^{n} V^{(p)}(\partial T^{(p)}/\partial x_1)$$

Applying Eq. (2a) to each component, one obtains the expression of the heat flux for each component, i.e.,

$$-q_i^{(p)} = K_{ij}^{(p)}[\partial T^{(p)}/\partial x_j] \qquad (p = 1, \ldots, n) \quad (i, j = 1, 2, 3) \tag{7}$$

Based on the previous two assumptions, the additional relations for the heat-flux and temperature gradient are obtained as

$$q_1 = q_1^{(p)} \quad (p = 1, 2, \ldots, n) \tag{8}$$

$$q_i = \sum_{p=1}^{n} V^{(p)} q_i^{(p)} \quad i = 2, 3 \tag{9}$$

$$\frac{\partial T}{\partial x_1} = \sum_{p=1}^{n} V^{(p)} \frac{\partial T^{(p)}}{\partial x_1} \tag{10}$$

$$\frac{\partial T}{\partial x_i} = \frac{\partial T^{(p)}}{\partial x_i} \qquad (p = 1, 2, \ldots, n) \quad (i = 2, 3) \tag{11}$$

The system of equations, i.e., Eqs. (7–11), represents $(6n+3)$ equations with $(6n+6)$ unknowns, namely $q_i^{(p)}$, $\partial T^{(p)}/\partial x_i$, q_i and $\partial T/\partial x_i$. Of these unknowns, $6n$ can therefore be eliminated from these equations to give three equations with six unknowns. The results after the elimination are as follows:

$$-q_1 = \frac{1}{\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]} \times \frac{\partial T}{\partial x_1} + \frac{\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]K_{12}^{(p)}}{\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]} \times \frac{\partial T}{\partial x_2} + \frac{\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]K_{13}^{(p)}}{\sum_{K=1}^{n}[V^{(p)}/K_{11}^{(p)}]} \times \frac{\partial T}{\partial x_3} \tag{12a}$$

$$-q_2 = \sum_{p=1}^{n} \frac{V^{(p)}K_{21}^{(p)}}{K_{11}^{(p)}\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]} \frac{\partial T}{\partial x_1} + \sum_{p=1}^{n}\left\{ V^{(p)}K_{22}^{(p)} + \frac{V^{(p)}K_{21}^{(p)}\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]K_{12}^{(p)}}{K_{11}^{(p)}\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]} - \frac{V^{(p)}K_{21}^{(p)}K_{12}^{(p)}}{K_{11}^{(p)}} \right\} \frac{\partial T}{\partial x_2} + \sum_{p=1}^{n}\left\{ V^{(p)}K_{23}^{(p)} + \frac{V^{(p)}K_{21}^{(p)}\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]K_{13}^{(p)}}{K_{11}^{(p)}\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]} - \frac{V^{(p)}K_{21}^{(p)}K_{13}^{(p)}}{K_{11}^{(p)}} \right\} \frac{\partial T}{\partial x_3} \tag{12b}$$

$$-q_3 = \sum_{p=1}^{n} \frac{V^{(p)}K_{31}^{(p)}}{K_{11}^{(p)}\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]} \frac{\partial T}{\partial x_1} + \sum_{p=1}^{n}\left\{ V^{(p)}K_{32}^{(p)} + \frac{V^{(p)}K_{31}^{(p)}\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]K_{12}^{(p)}}{K_{11}^{(p)}\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]} - \frac{V^{(p)}K_{31}^{(p)}K_{12}^{(p)}}{K_{11}^{(p)}} \right\} \frac{\partial T}{\partial x_2} + \sum_{p=1}^{n}\left\{ V^{(p)}K_{33}^{(p)} + \frac{V^{(p)}K_{31}^{(p)}\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]K_{13}^{(p)}}{K_{11}^{(p)}\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]} - \frac{V^{(p)}K_{31}^{(p)}K_{13}^{(p)}}{K_{11}^{(p)}} \right\} \frac{\partial T}{\partial x_3} \tag{12c}$$

These equations relate q_i with $\partial T/\partial x_i$ for the composite. Since the Fourier's law given in Eq. (2a) will apply for the smoothed composite, one obtains, by comparing the coefficients appearing in Eq. (2a) with those in Eq. (12), the conductivity K_{ij} of the composite in terms of the conductivities of the components $K_{ij}^{(p)}$ as

$$K_{ij} = \frac{1}{a}\sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]K_{ij}^{(p)} \qquad \begin{matrix} i = 1, & j = 1, 2, 3 \\ j = 1, & i = 1, 2, 3 \end{matrix}$$

$$K_{ij} = \sum_{p=1}^{n} V^{(p)}\left[K_{ij}^{(p)} + \frac{K_{i1}^{(p)}}{aK_{11}^{(p)}}\sum_{p=1}^{n}\frac{V^{(p)}}{K_{11}^{(p)}}K_{1j}^{(p)} - \frac{K_{i1}^{(p)}K_{1j}^{(p)}}{K_{11}^{(p)}}\right]; \qquad i, j = 2, 3 \tag{13}$$

where

$$a = \sum_{p=1}^{n}[V^{(p)}/K_{11}^{(p)}]$$

It is observed from Eq. (13) that, for isotropic layer where $K_{ij}^{(p)} = \delta_{ij}K^{(p)}$, the conductivities of the composite are reduced to

$$K_{11} = 1\Big/\sum_{p=1}^{n}[V^{(p)}/K^{(p)}] \tag{14a}$$

$$K_{22} = \sum_{p=1}^{n} V^{(p)}K^{(p)} = K_{33} \tag{14b}$$

These relations can be found from standard textbooks.

Parallel to the development of the conductivity matrix, we may apply the same procedures to obtain the elements of the resistivity matrix of the composite. To this end, we first apply Eq. (3) to each layer. The resulting equations are combined with Eqs. (8–11) to form a system of equations. Upon eliminating $(6n)$ unknowns from the system, three equations for q_i corresponding to Eq. (12) are obtained. In this way, we get the elements of the resistivity matrix that are given in the Appendix. As a final check, if we multiply this resistivity matrix with the conductivity matrix developed earlier, an identity matrix results, which is consistent with the statement described in Eq. (4).

With the K_{ij} of the composite known, one can solve for temperature, temperature gradient $\partial T/\partial x_i$ and heat flux q_i from the field differential equations (2b) with its boundary conditions and the general Fourier's equations (2a). The solution thus obtained is approximate since the composite under consideration is a smoothed homogeneous anisotropic one. Because the composite is actually made of layers, the information of temperature gradient $\partial T^{(p)}/\partial x_i$ and heat flux $q_i^{(p)}$ for each layer would be of interest. To obtain such information, we consider a system of equations including Eqs. (7, 8, and 11), which are $(6n)$ equations. With $\partial T/\partial x_i$ and q_i just found, this system will have $(6n)$ unknowns, and $\partial T^{(p)}/\partial x_i$ and $q_i^{(p)}$ can then be solved. These are

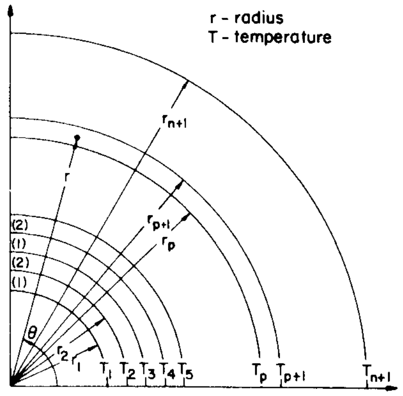

Fig. 4 Cylindrical sector of a bilamina composite.

$$q_1^{(p)} = q_1 \qquad (p = 1, 2, \ldots, n)$$
$$\partial T^{(p)}/\partial x_i = \partial T/\partial x_i \qquad (i = 2, 3; p = 1, 2, \ldots, n)$$
$$\frac{\partial T^{(p)}}{\partial x_1} = \frac{-1}{K_{11}^{(p)}}\left(q_1 + K_{12}^{(p)}\frac{\partial T}{\partial x_2} + K_{13}^{(p)}\frac{\partial T}{\partial x_3}\right) \quad (p = 1, 2, \ldots, n)$$
$$q_i^{(p)} = K_{ij}^{(p)}[\partial T^{(p)}/\partial x_j] \quad (i = 2, 3; p = 1, 2, \ldots, n) \tag{15}$$

It has to be mentioned that the derivation of the expressions for K_{ij} is based on the assumption that each layer is very thin. The element shown in Fig. 2 thus is equivalent to the small element in deriving the classical isotropic Fourier's heat conduction equations. To the limit, when this element is shrunk into a point, Eq. (2a) with K_{ij} given by Eq. (13) represents the material property at a "point." It is well known that constitutive relations are point functions and are independent on the coordinate system, as long as it is orthogonal. The results thus obtained will equally apply for cylindrical and spherical coordinates although the present derivation is for Cartesian coordinates.

Sample Calculations

To demonstrate the validity of the calculated thermal conductivity K_{ij} and the procedure for solving heat conduction problems for a composite described in the foregoing, we consider a simple problem where an exact solution exists. This is the case of the cylindrical bi-lamina composite material shown in Fig. 4, the repeated layers denoted by "1" and "2" in the figure being anisotropic. If the temperature distribution is specified on the inner and the outer surfaces, the heat flow will depend on both r and θ coordinates. If the constant temperature gradient in θ direction is imposed, the problem can be solved exactly. The approximate method is based on the consideration that the composite material is homogeneous and anisotropic. In this way, one may compute the thermal conductivities K_{ij} for the composite material in terms of the components conductivities using the relations developed earlier. Temperature distribution and heat flow for the unsmoothed composite are then calculated. The thermal conductivities of the individual components $K_{ij}^{(p)}$ $(p = 1, 2)$ will be involved in these solutions. With both solutions available, a comparison of their results will finally be presented.

Approximate Method

For a homogeneous and anisotropic material with heat flow along r and θ coordinates only, the general Fourier Law assumes the form

$$-q_r = K_{11}(\partial T/\partial r) + K_{12}(\partial T/r\,\partial\theta)$$
$$-q_\theta = K_{21}(\partial T/\partial r) + K_{22}(\partial T/r\,\partial\theta) \tag{16}$$

The governing differential equation for steady heat conduction with constant K_{ij} in the present case is

$$\frac{K_{11}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{K_{22}}{r^2}\frac{\partial^2 T}{\partial\theta^2} + \frac{1}{r}(K_{12}+K_{21})\frac{\partial^2 T}{\partial r\,\partial\theta} = 0 \tag{17}$$

Consider the case where a constant temperature gradient in θ-direction is imposed, i.e.,

$$\partial T/\partial\theta = -C \tag{18}$$

Equation (17) then becomes

$$(1/r)(\partial/\partial r)[r(\partial T/\partial r)] = 0 \tag{19}$$

The boundary conditions are

$$T = T_1(\theta) \quad \text{at} \quad r = r_1$$
$$T = T_{n+1}(\theta) \quad \text{at} \quad r = r_{n+1} \tag{20}$$

where n is the number of layers.

Let us next introduce dimensionless quantities in the form

$$\phi = (T - T_1)/(T_{n+1} - T_1)$$
$$\bar{r} = r/r_1$$
$$\bar{C} = C/(T_{n+1} - T_1)$$
$$\bar{q}_r = [q_r r_1/(T_{n+1} - T_1)][1/K_{11}^{(1)}]$$
$$\bar{q}_\theta = [q_\theta r_1/(T_{n+1} - T_1)][1/K_{11}^{(1)}]$$
$$\bar{K}_{ij} = K_{ij}/K_{11}^{(1)} \quad (i, j = 1, 2, 3) \tag{21}$$

The solutions of the heat equation (19) with the boundary conditions (20) and the heat fluxes obtained from Eqs. (16) expressed in the dimensionless form are

$$\phi = \ln\bar{r}/\ln\bar{r}_{n+1}$$
$$\bar{q}_r = -(\bar{K}_{11}/r)(1/\ln\bar{r}_{n+1}) + (\bar{K}_{12}/r)\bar{C} \tag{22}$$
$$\bar{q}_\theta = -(\bar{K}_{21}/\bar{r})(1/\ln\bar{r}_{n+1}) + (\bar{K}_{22}/\bar{r})\bar{C}$$

K_{ij}'s appearing in these equations are calculated from Eq. (13) in terms of the conductivities of the components. Equation (22) is therefore referred to as approximate solution for the smeared composite.

Exact Solution

The governing equations corresponding to Eqs. (16–20) are written for kth layer of the composite material, $r_k < r < r_{k+1}$ $(k = 1, 2, \ldots, n)$ as follows:

$$-q_r^{(p)} = K_{11}^{(p)}(\partial T/\partial r) + K_{12}^{(p)}(\partial T/r\,\partial\theta)$$
$$-q_\theta^{(p)} = K_{21}^{(p)}(\partial T/\partial r) + K_{22}^{(p)}(\partial T/r\,\partial\theta) \tag{23}$$
$$\partial T/\partial\theta = -C \tag{24}$$
$$(\partial/\partial r)[r(\partial T/\partial r)] = 0 \tag{25}$$

The interfacial conditions for the kth layer is given as

$$r = r_k \qquad T = T_k$$
$$r = r_{k+1} \qquad T = T_{k+1} \tag{26}$$

The solutions of Eqs. (23–26) have a form similar to Eqs. (22); i.e., for the kth layer

$$\eta = \frac{\ln\bar{r}/\bar{r}_k}{\ln\bar{r}_{k+1}/\bar{r}_k}$$
$$\bar{q}_r^{(p)} = -\frac{K_{11}^{(p)}}{\bar{r}}(\bar{T}_{k+1} - \bar{T}_k)\frac{1}{\ln(\bar{r}_{k+1}/\bar{r}_k)} + \frac{\bar{K}_{12}^{(p)}}{\bar{r}}\bar{C} \tag{27}$$
$$\bar{q}_\theta^{(p)} = -\frac{\bar{K}_{21}^{(p)}}{\bar{r}}(\bar{T}_{k+1} - \bar{T}_k)\frac{1}{\ln(\bar{r}_{k+1}/\bar{r}_k)} + \frac{\bar{K}_{22}^{(p)}}{\bar{r}}\bar{C}$$

where

$$\eta = (T - T_k)/(T_{k+1} - T_k)$$
$$p = 1 \quad \text{for} \quad k = 1, 3, 5, \ldots, n-1$$
$$= 2 \quad \text{for} \quad k = 2, 4, 6, \ldots, n$$
$$\bar{K}_{ij}^{(p)} = K_{ij}^{(p)}/K_{11}^{(1)} \quad \text{and} \quad \bar{T} = (T - T_1)/(T_{n+1} - T_1)$$

The interfacial temperatures $\bar{T}_{k+1}$ and $\bar{T}_k$ appearing in the solution (27) can be obtained by matching the heat fluxes in radial direction at each interface.

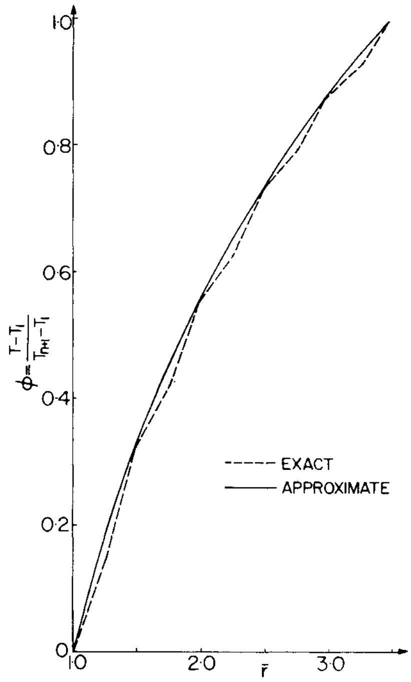

Fig. 5 Dimensionless temperature distribution.

Results

Let us consider a hypothetical cylindrical sector with ten equal layers (i.e., $n = 10$). For the sake of convenience, the numerical values of the conductivities of the components, the radius of the sector and the thickness of the layers are taken as follows: $K_{11}^{(1)} = 100$ Btu/ft hr F; $K_{21}^{(1)} = \frac{1}{4}K_{11}^{(1)}$; $K_{11}^{(2)} = 50$ Btu/ft hr F; $K_{21}^{(2)} = \frac{1}{4}K_{11}^{(2)}$; $r_1 = 1$ in.; $L_1 = L_2 = \frac{1}{4}$ in.; and $\bar{C} = 1$; where L_1 and L_2 are referred to the thickness of the layers for the first component and second component, respectively. From the given values of the thermal conductivity $K_{ij}^{(p)}$ for each component, the thermal conductivity K_{ij} for the composite is calculated from Eq. (13). With this information available, we can obtain the approximate and the exact solutions described in the foregoing. The results of these solutions are presented in Figs. 5–7. Figure 5 shows the dimensionless temperature ϕ vs dimensionless radius $\bar{r}$. The dash and solid curves represent respectively the results of the exact solution and approximate solution. Their comparison is satisfactory. The dimensionless temperatures from approximate solution at the locations $r/r_0 = 1.5$, 2, 2.5, and 3 are very close to those from exact solutions although they are not precisely the same. Their differences, however, are within the scale of the plot. The comparison of dimensionless heat flux in r direction is given in Fig. 6. Their differences are very small. The heat flux in θ direction q_θ, obtained from the exact solution has discontinuity at the interfaces as shown in Fig. 7 but it is continuous when obtained from the approximate solution for the smoothed composite. In order to compare both solutions for the unsmoothed composite, we calculated $q_\theta^{(p)}$ from the second of Eq. (23), where $\partial T/\partial r$ appearing in this equation is obtained from the temperature distribution of the approximate solution. This computation procedure has been discussed in the latter part of the analysis. The solid curves shown in Fig. 7 represent the results of this calculation. The comparison of these two solutions for the heat flux in θ direction is satisfactory.

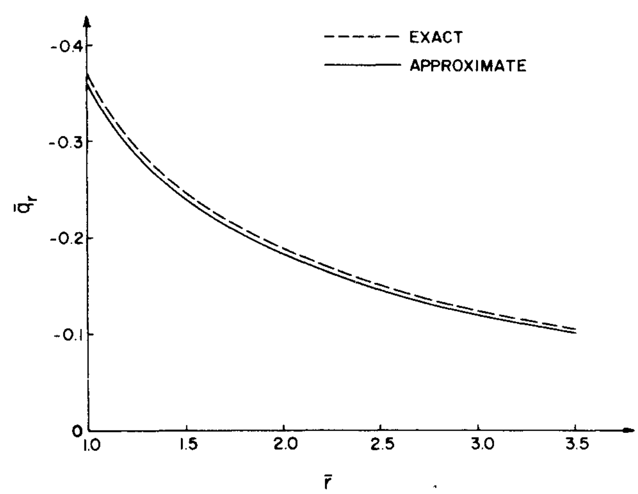

Fig. 6 Distribution of heat flux in radial direction.

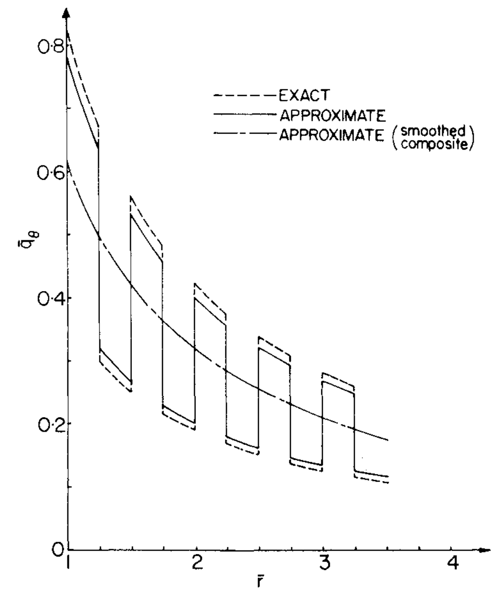

Fig. 7 Distribution of heat flux in θ-direction.

It has to be noted that the present method of apparent conductivity does not allow one to estimate the errors for both temperature distribution and heat flux. In this sample calculation, layers of $\frac{1}{4}$ in. thickness were used. The derivation of the expressions for K_{ij}, as mentioned earlier, is based on the assumption that each layer is very thin; i.e., the error is smaller for thinner layers. The actual composite used today having the thickness of the layer in the order of 10μ–100μ, will give less error as compared to that given in the sample calculation. The method is therefore fairly accurate and fast.

Conclusion

A lamination theory has been proposed to obtain the thermal conductivities of a small element of a composite material in terms of its components' conductivities and to solve the boundary value conduction problem of the composite. To demonstrate the method, a sample calculation employing a bilamina cylindrical sector is performed. The result of the calculation including the temperature distribution and the heat fluxes based on the computed conductivities compares satisfactorily with that of the exact solution. The interfacial resistances and the associated radiation contribution, however, have not been taken into account in this work.

Appendix

The conductivity tensor of the smeared composite has been given in Eq. (13) of the text. The resistivity tensor appears in a more complicated form; it is presented as follows:

$$R_{ij} = \sum_{p=1}^{n} \frac{V^{(p)}[R_{11}{}^{(p)}A^{(p)} + R_{12}{}^{(p)}C^{(p)} + R_{13}{}^{(p)}B^{(p)}]}{A^{(p)}} - \frac{1}{d}\left[g \sum_{p=1}^{n} \frac{V^{(p)}D^{(p)}}{A^{(p)}} + f \sum_{p=1}^{n} \frac{V^{(p)}E^{(p)}}{A^{(p)}}\right] \quad i,j = 1$$

$$R_{ij} = \frac{1}{d}\left[\sum_{p=1}^{n} \frac{V^{(p)}R_{2j}{}^{(p)}}{A^{(p)}} \times \sum_{p=1}^{n} \frac{V^{(p)}D^{(p)}}{A^{(p)}} + \sum_{p=1}^{n} \frac{V^{(p)}R_{3j}{}^{(p)}}{A^{(p)}} \times \sum_{p=1}^{n} \frac{V^{(p)}E^{(p)}}{A^{(p)}}\right] \quad i = 1;\ j = 2,3$$

$$R_{ij} = -\frac{1}{d} \sum_{p=1}^{n} \frac{V^{(p)}R_{ij}{}^{(p)}}{A^{(p)}} \qquad i,j = 2,3$$

$$R_{ij} = \frac{1}{d}\left[\sum_{p=1}^{n} \frac{V^{(p)}R_{i3}{}^{(p)}}{A^{(p)}} \times \sum_{p=1}^{n} \frac{V^{(p)}B^{(p)}}{A^{(p)}} + \sum_{p=1}^{n} \frac{V^{(p)}R_{i2}{}^{(p)}}{A^{(p)}} \times \sum_{p=1}^{n} \frac{V^{(p)}C^{(p)}}{A^{(p)}}\right] \quad j = 1;\ i = 2,3$$

where

$$A^{(p)} = R_{23}{}^{(p)}R_{32}{}^{(p)} - R_{33}{}^{(p)}R_{22}{}^{(p)}$$
$$B^{(p)} = R_{31}{}^{(p)}R_{22}{}^{(p)} - R_{21}{}^{(p)}R_{32}{}^{(p)}$$
$$C^{(p)} = R_{21}{}^{(p)}R_{33}{}^{(p)} - R_{23}{}^{(p)}R_{31}{}^{(p)}$$
$$D^{(p)} = R_{12}{}^{(p)}R_{33}{}^{(p)} - R_{13}{}^{(p)}R_{32}{}^{(p)}$$
$$E^{(p)} = R_{13}{}^{(p)}R_{32}{}^{(p)} - R_{12}{}^{(p)}R_{23}{}^{(p)}$$

$$d = \sum_{p=1}^{n} \frac{V^{(p)}R_{33}{}^{(p)}}{A^{(p)}} \times \sum_{p=1}^{n} \frac{V^{(p)}R_{22}{}^{(p)}}{A^{(p)}} - \sum_{p=1}^{n} \frac{V^{(p)}R_{32}{}^{(p)}}{A^{(p)}} \times \sum_{p=1}^{n} \frac{V^{(p)}R_{23}{}^{(p)}}{A^{(p)}}$$

$$f = \sum_{p=1}^{n} \frac{V^{(p)}R_{32}{}^{(p)}}{A^{(p)}} \times \sum_{p=1}^{n} \frac{V^{(p)}C^{(p)}}{A^{(p)}} + \sum_{p=1}^{n} \frac{V^{(p)}R_{33}{}^{(p)}}{A^{(p)}} \times \sum_{p=1}^{n} \frac{V^{(p)}B^{(p)}}{A^{(p)}}$$

$$g = \sum_{p=1}^{n} \frac{V^{(p)}R_{23}{}^{(p)}}{A^{(p)}} \times \sum_{p=1}^{n} \frac{V^{(p)}B^{(p)}}{A^{(p)}} + \sum_{p=1}^{n} \frac{V^{(p)}R_{22}{}^{(p)}}{A^{(p)}} \times \sum_{p=1}^{n} \frac{V^{(p)}C^{(p)}}{A^{(p)}}$$

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